DOE/ER/53223-32
MF-114

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Mathematical Sciences

Magneto-Fluid Dynamics Division

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U.S. Department of Energy
Grant No. DE-FG02-86ER53223

March 1987

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Printed in U.S.A.

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Springfield, VA 22161

PLASMA DIFFUSION AND RELAXATION DUE TO LOW-LEVEL TURBULENCE*

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ABSTRACT

A theory of low-level turbulence in compressible plasmas is presented. It is characterized by a separation of scales between the mean fields and the fluctuations, which are of low amplitude. Use of conservation laws of the system enables the derivation of important factors in the functional form of the turbulent diffusivity coefficients. In particular, it is shown that turbulent heat conductivity, like its molecular counterpart, transports heat in the direction opposite to the mean temperature gradient. The turbulent resistivity, on the other hand, acts non-classically and causes flattening of the parallel current profile so that the plasma tends to approach a "Taylor state". The case of planar, viscous, fluid flow is treated in detail as a special example. It is shown that turbulent viscosity in the low-level turbulence case simply causes enhancement of the molecular viscosity.

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*Presented at the Fifth International Beer-Sheva Seminar on MHD Flows and Turbulence, March 2-6, 1987, Jerusalem, Israel. To appear in the AIAA Progress Series.

I. INTRODUCTION

The subject of turbulence, namely, the description of flows which vary rapidly in space and time, does not easily lend itself to either analytical or numerical methods. Indeed, many of the statistical methods[1,2] used by physicists and engineers are not established on a rigorous basis and are, in fact, known to yield incorrect answers under some circumstances which are, unfortunately, not always definable. There is an exception to this rule and, quite remarkably, it has escaped attention until very recently. This is the area of low-level turbulence.

By the term "low-level turbulence" we mean that the ratio of amplitudes of the fluctuations to the mean flow is small, as well as that there is a clear separation of both temporal and spatial scales between these two components of the flow: the fluctuations vary on a faster time scale and a shorter spatial scale than the mean flow. These assumptions make possible the use of multiscale asymptotic methods which lead to simplified equations and allow, in some cases, the derivation of an exact expression for the turbulent "eddy" diffusivity. An example is given in Section II. In this context we mention the work of McLaughlin et al.[3] who use such methods mostly for fluctuations of large amplitude. This limits the extent of the analytical results that can be derived.

Low-level turbulence may not be too pressing a subject in fluid dynamics, but it is precisely the natural turbulence problem in plasma confinement studies. Experience shows that the confinement of a hot plasma by a magnetic field is a very delicate matter. If the plasma undergoes large fluctuations, confinement is lost roughly within a fluctuation period. By definition, therefore, fluctuations in a confined plasma are of small amplitude. The separation of scales too arises naturally when one identifies the origin of the fluctuations in the myriad of instabilities affecting the plasma, many of which tend to be localized about a "resonant" magnetic field line.[4,5] Perhaps this recognition gave rise to some works[6-8] which take advantage of disparate scales in the plasma dynamics. Their results are still too cumbersome to offer an understanding of the turbulent diffusivity of the plasma.

A most intriguing phenomenon which, we believe, is explained by the effect of low-level turbulence is the spontaneous reversal of the toroidal magnetic field in Reversed-Field Pinch (RFP) experiments.[9] According to the theory of plasma relaxation due to Taylor[10] (and recently reviewed by him[11]), a slightly resistive plasma tends to reach an equilibrium state of minimum magnetic energy subject to the conservation of its magnetic helicity, thus producing a force-free state, now known as a "Taylor state", with $\nabla\times\underset{\sim}{B} = \lambda_o\underset{\sim}{B}$, where $\underset{\sim}{B}$ is the magnetic field and λ_o is a constant. Solutions of this equation approximate general features of observed magnetic configurations in the RFP, including toroidal field reversal[9], but the theory does not provide a dynamical description of the relaxation process. A numerical calculation due to Strauss,[12] and based on a magnetohydrodynamic (MHD) description of the plasma with disparate scales taken into account, does show such an effect. Our analytical work with Bhattacharjee on low-level turbulence in a plasma,[13,14] successfully determined an explicit expression for the eddy resistivity (which involves a fourth derivative of $\underset{\sim}{B}$), along with its dependence on the mean fields.

Various other considerations[15-17] lead to similar, if less precise, results. It was demonstrated[13,14] that a plasma under the influence of such eddy resistivity does tend to approach a Taylor state. It was also shown[18] that the plasma need not obey Taylor's energy principle. If anything, it follows some entropy production principle.

In this paper we extend our earlier work,[13,14] in which resistivity was considered to be the only diffusive effect, to plasma with finite resistivity and heat conductivity. This also means that, as in Ref. 14, we deal here with a compressible plasma, a case which is almost never treated in the literature of the analogous question of the "dynamo" problem.[19] Rather than getting an exact expression for the "eddy" heat conductivity, we are content here with producing an expression for it in terms of means of fluctuating quantities, and with exploring its general structure. It turns out to be more similar to the molecular heat conductivity than eddy resistivity is to its molecular counterpart. This is done in Sections IV and V, after a review of earlier results in Section III. But first we start with a demonstration the multiscale method and its effectiveness in an easier problem: two-dimensional turbulent flow and the question of eddy viscosity. In this special case, an expression for the eddy viscosity can be explicitly determined if the level of turbulence is low.

II. TWO-DIMENSIONAL FLUID FLOW

We consider the planar motion of an incompressible fluid, which obeys the Navier-Stokes equations

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v} + \frac{1}{\rho}\nabla p = \nu\Delta\underset{\sim}{v} \tag{1}$$

$$\nabla\cdot\underset{\sim}{v} = 0 \ . \tag{2}$$

Here $\underset{\sim}{v}$, p and ρ are the velocity, pressure and the constant density, respectively, and ν is the constant viscosity. The curl of Eq. (1) yields

$$\frac{\partial \omega}{\partial t} + \underset{\sim}{v}\cdot\nabla\omega = \nu\Delta\omega \ , \tag{3}$$

where ω is the component of the vorticity $\nabla\times\underset{\sim}{v}$ normal to the plane.

Let us write every quantity as the sum of a mean part, denoted by a subscript o, and a fluctuating part, denoted by a subscript 1. Thus, $\underset{\sim}{v} = \underset{\sim}{v}_o + \underset{\sim}{v}_1$, where $\underset{\sim}{v}_o = \langle\underset{\sim}{v}\rangle$ and $\langle\underset{\sim}{v}_1\rangle = 0$, and the brackets $\langle\ \rangle$ denote either an ensemble average or an average over the spatial and temporal scales of the fluctuations. We now make the following assumptions: (1) the mean state is near steady state and evolves slowly on a viscous time scale, while it varies spatially on the scale a of the system, i.e., $a|\underset{\sim}{v}_o| = O(1)$, $a^2\partial/\partial t = O(\nu)$, $a|\nabla| = O(1)$; (2) the fluctuations vary on a shorter space-scale and a faster time-scale, i.e., $a^2\partial/\partial t \gg O(\nu)$, $a|\nabla| \gg O(1)$; (3) the fluctuation level is so low that $|\omega_1| \ll |\omega_o|$ and $\nabla\omega_1$ is at most order 1, i.e., $a|\nabla\omega_1|/\omega_o = O(1)$. In the following we will consider all quantities to have been nondimensionalized; spatial variables scaled with a, and velocities scaled with U, where U is a typical mean velocity.

Averaging Eq. (3) yields

$$\frac{\partial \omega_o}{\partial t} + \underset{\sim}{v}_o \cdot \nabla \omega_o + F = \nu \Delta \omega_o \ , \qquad F \equiv \text{div} \ \langle \omega_1 \underset{\sim}{v}_1 \rangle \ . \tag{4}$$

The term F embodies the effect of the fluctuations on the mean flow and, according to our ordering, should be small. To leading order the mean flow is in steady state, satisfying $\underset{\sim}{v}_o \cdot \nabla \omega_o = 0$. Thus, the evolution may be viewed as taking place through a sequence of near-equilibrium states. This is analogous to the Grad-Hogan theory of slow plasma diffusion.[20] For simplicity let us assume that each such steady state generates a nested family of streamlines as in Fig. 1. To study F, let us write Eq. (4) in the form

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \omega_o^2\right) + \text{div} \ (\omega_o \underset{\sim}{v}_o - \nu \omega_o \nabla \omega_o) + \omega_o F = -\nu |\nabla \omega_o|^2 \ . \tag{5}$$

We can generate another such equation by writing Eq. (3) as

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \omega^2\right) + \text{div} \ \left(\frac{1}{2} \omega^2 \underset{\sim}{v} - \nu \omega \nabla \omega\right) = -\nu |\nabla \omega|^2 \ , \tag{6}$$

and averaging. The result, to order ν, is

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \omega_o^2\right) + \text{div} \ \left(\frac{1}{2} \omega_o^2 \underset{\sim}{v}_o - \nu \omega_o \nabla \omega_o\right) + \text{div} \ \langle \omega_o \omega_1 \underset{\sim}{v}_1 + \frac{1}{2} \omega_1^2 \underset{\sim}{v}_o \rangle$$
$$= -\nu |\nabla \omega_o|^2 - \nu \langle |\nabla \omega_1|^2 \rangle \ . \tag{7}$$

Note that the averaging is taken under the derivatives, and that the term $\partial\langle\omega_1^2\rangle/\partial t$ is $o(\nu)$. Comparing Eqs. (5) and (7), we have to $O(\nu)$

$$\omega_o F = \nu\langle|\nabla\omega_1|^2\rangle + \mathrm{div}\,\langle\omega_o\omega_1\underset{\sim}{v}_1 + \frac{1}{2}\,\omega_1^2\underset{\sim}{v}_o\rangle \ . \tag{8}$$

Writing $F = \nabla\cdot\underset{\sim}{\xi}$, where $\underset{\sim}{\xi} = \langle\omega_1\underset{\sim}{v}_1\rangle$, we have (if there is no flow across the boundary) $\int F d^2x = 0$, while from (8) we have $\int\omega_o F d^2x \geq 0$. The integrals here are taken over the flow region. Integrating by parts, we get

$$\int (\underset{\sim}{\xi}\cdot\nabla\omega_o)d^2x \leq 0 \ . \tag{9}$$

In analogy with the Grad-Hogan theory,[20,21] we consider the geometry of closed streamlines of the mean flow, label each curve by A, the area it encloses, $A = A(\underset{\sim}{x})$, and write $d^2x = dA\,d\ell/|\nabla A|$, where $d\ell$ is a length element along the closed curve. Thus, Eq. (9) can be rewritten as

$$\int dA\,\frac{d\omega_o}{dA}\oint_A \xi_n d\ell \leq 0 \ . \tag{10}$$

Here ξ_n is the component normal to the curve, the subscript A indicates the particular curve with label A, and we have used the fact that, to $O(1)$, $\omega_o = \omega_o(A)$. Now, if the correlation of the turbulence across mean streamlines is negligible, as is the case when the multiscale method applies, then inequality (10) must be satisfied separately on each curve. We thus find the following property of the eddy viscosity term:

$$\oint_A \xi_n d\ell = -K^2 \frac{d\omega_o}{dA} , \qquad (11)$$

where K^2 is a non-negative function. This conclusion will shortly be verified in the example given below. It is worthwhile to point out that similar arguments could have led to the conclusion that $\oint F d\ell/|\nabla A|$ should be proportional to ω_o, or to other conclusions along similar lines. We emphasize that conclusion (11) relates to $\underset{\sim}{\xi}$ itself, and not to its derivatives. For turbulence with short correlation length, derivatives of $\underset{\sim}{\xi}$ can be much larger than $\underset{\sim}{\xi}$ and an ordering scheme leading to Eq. (8) might fail.

The implication of (11) can be seen if we take the (microcanonical) average[20,21] of Eq. (4), that is, integrate it $d\ell/|\nabla A|$. Using incompressibility, and the fact that for any f

$$\oint_A f \frac{d\ell}{|\nabla A|} = \frac{d}{dA} \int_A f d^2x ,$$

where the subscript A on the right-hand-side indicates integration within the curve A, we find

$$\frac{\partial}{\partial t} \omega_o(A,t) + \frac{\partial}{\partial A} \oint_A \xi_n d\ell = \nu \frac{\partial}{\partial A} \{\frac{\partial \omega_o}{\partial A} \oint_A |\nabla A| d\ell\} , \qquad (12)$$

where the first term follows from non-trivial averaging.[21] Equation (12) is a one-dimensional diffusion equation for $\omega_o(A,t)$, with respect to the given geometry of streamlines. (Of course, the streamlines themselves also change at the same time, and the slow evolution problem needs to be solved iteratively.[20,21] Combining Eqs. (11) and (12), we get

$$\frac{\partial}{\partial t}\,\omega_o(A,t) = \frac{\partial}{\partial A}\,\{[\nu \oint_A |\nabla A|d\ell + K^2]\,\frac{\partial \omega_o}{\partial A}\} \;. \qquad (13)$$

The K^2 term, which may depend on the shape of the streamlines, expresses the enhancement of diffusion by eddy viscosity. Its effect, in this case, appears to be similar to that of the molecular viscosity. Using similar arguments, we will see in the next section that the same is not true for the electrical resistivity.

It will be comforting to derive property (11) directly from the dynamical equations of the fluctuations. For this we need to adopt a more specific ordering (which is why we preferred to derive the more general result first). We use the simplest geometry possible and assume the mean streamlines to form, to leading order, concentric circles. In polar coordinates r,θ, we have $\omega_o = \omega_o(r) + O(\nu)$. Subtracting Eq. (4) from (3), we get the evolution equation of the fluctuations,

$$\frac{\partial \omega_1}{\partial t} + \underset{\sim}{v}_o\cdot\nabla\omega_1 + \underset{\sim}{v}_1\cdot\nabla\omega_o + \underset{\sim}{v}_1\cdot\nabla\omega_1 - \mathrm{div}\,\langle\omega_1\underset{\sim}{v}_1\rangle = \nu\Delta\omega_1 \;. \qquad (14)$$

We assume the fluctuations to vary rapidly across mean streamlines, but to have O(1) variation along them. Introducing the smallness parameter

δ, we use the small scale variables $\rho = r/\delta$, $\tau = \delta^2 t$, and make the multiscale ansatz

$$\underset{\sim}{v} = \underset{\sim}{v}_1(\rho,\tau;r,\theta,t) \ , \tag{15}$$

where we take $\nu = O(\delta^4)$. Note that in Eq. (14) we now have to replace $\partial/\partial t \rightarrow \partial/\partial t + \delta^2\partial/\partial\tau$, $\partial/\partial r \rightarrow \partial/\partial r + \delta^{-1}\partial/\partial\rho$. The order of the perturbed velocity components (indicated by subscripts) is taken to be

$$v_{1r} = \delta^3 v_r^{(3)} + O(\delta^4) \ , \qquad v_{1\theta} = \delta^2 v_\theta^{(2)} + O(\delta^3)$$

$$\omega_1 = \delta\omega^{(1)} + \delta^2\omega^{(2)} + \delta^3\omega^{(3)} + O(\delta^4) \ ,$$

where $\omega^{(1)} = \partial v_\theta^{(2)}/\partial\rho$, but higher order terms of ω_1 depend on higher order terms of the velocity. Up to $O(\delta^3)$, Eq. (14) reads

$$\delta^{-3}\left(\frac{\partial\omega_1}{\partial t} + \underset{\sim}{v}_o\cdot\nabla\omega_1\right) + \frac{\partial\omega^{(1)}}{\partial\tau} + \omega_o' v_r^{(3)} + \left(v_r^{(3)}\frac{\partial}{\partial\rho} + \frac{1}{r}v_\theta^{(2)}\frac{\partial}{\partial\theta}\right)\omega^{(1)} = \nu\frac{\partial^2\omega^{(1)}}{\partial\rho^2} \ . \tag{16}$$

Here $\omega_o' = d\omega_o/dr$, and we notice that the term div $\langle\omega_1\underset{\sim}{v}_1\rangle$ in Eq. (14), which is of the order of $\omega_1 v_{1r} = O(\delta^4)$, has dropped out. The terms in the first parentheses in Eq. (16) are sums of terms ranging from $O(\delta)$ to $O(\delta^3)$. Thus, to $O(\delta)$, we have

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}_o\cdot\nabla\right)\omega^{(1)} = 0 \ , \tag{17}$$

and the same holds for $\omega^{(2)}$. Equation (17) implies that $\omega^{(1)}$ is advected circularly by the mean flow, as should be expected.

It is useful to replace θ by the angular Lagrangian coordinate of the mean flow, $\theta_o = \theta - \Omega_o(r)t$, where $\Omega_o(r)$ is the rotational frequency of the mean flow on the curve $r = \text{const}$. The corresponding change in Eq. (16) is $\partial/\partial\theta \to \partial/\partial\theta_o$ and $\partial/\partial t + \underline{v}_o\cdot\nabla \to \partial/\partial t$. At $O(\delta^3)$, the structure of Eq. (16) is

$$\frac{\partial\omega^{(3)}}{\partial t} = f(\rho,\tau;r,\theta_o) \ , \qquad (18)$$

where the independence of the right-hand-side on t follows from Eq. (17) and the fact that $v_\theta^{(2)}$ and $v_r^{(3)}$ are determined from $\omega^{(1)}$ by using the definition of $\omega^{(1)}$ and the divergence-free condition,

$$\frac{\partial v_r^{(3)}}{\partial\rho} + \frac{1}{r}\frac{\partial v_\theta^{(2)}}{\partial\theta_o} = 0 \ , \qquad \frac{\partial v_\theta^{(2)}}{\partial\rho} = \omega^{(1)} \ . \qquad (19)$$

The solution $\omega^{(3)}$ of Eq. (18) grows linearly in t, unless $f = 0$. We thus require

$$\frac{\partial\omega^{(1)}}{\partial\tau} + \omega_o' v_r^{(3)} + \left(v_r^{(3)}\frac{\partial}{\partial\rho} + \frac{1}{r}v_\theta^{(2)}\frac{\partial}{\partial\theta_o}\right)\omega^{(1)} = \nu\frac{\partial^2\omega^{(1)}}{\partial\rho^2} \ . \qquad (20)$$

Such a non-secularity condition is common in multiscale analyses. Solutions of Eqs. (19),(20) depend on the independent variables τ, ρ and θ_o, as well as on the three parameters ν, r and ω_o'. The parameters can be eliminated by the following scaling transformation (which effectively sets them to 1):

$$\rho \rightarrow \left(\frac{r\nu}{\omega_o'}\right)^{1/4}\rho\ , \qquad \tau \rightarrow \left(\frac{r}{\nu\omega_o'}\right)^{1/2}\tau\ , \qquad \omega^{(1)} \rightarrow (r\nu\omega_o'^{3})^{1/4}\omega^{(1)}$$

$$v_r^{(3)} \rightarrow \left(\frac{1}{r}\,\nu^3\omega_o'\right)^{1/4}v_r^{(3)}\ , \qquad v_\theta^{(2)} \rightarrow (r\nu\omega_o')^{1/2}v_\theta^{(2)}\ . \tag{21}$$

The eddy viscosity term can now be determined to order $\nu = O(\delta^4)$,

$$\langle\omega_1 v_{1r}\rangle = -C\nu\omega_o'\ , \qquad C > 0\ , \tag{22}$$

where C is some universal constant. Its positivity follows from Eq. (9) or, more directly, by multiplying Eq. (20) by $\omega^{(1)}$ and averaging over the independent variables. In this simple geometry, Eq. (4) implies that the eddy viscosity simply equals $C\nu$. A similar relation, up to a dependence of C on the geometry of mean streamlines, is expected in the more general two-dimensional flows.

III. TURBULENT DIFFUSION IN INCOMPRESSIBLE PLASMAS

In this section we review some results[13,14] concerning the eddy resistivity of incompressible plasmas. The next section develops additional results for the compressible case, including an expression for the eddy heat conductivity. The plasma is assumed to obey the resistive MHD equations

$$\rho\left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla\right)\underset{\sim}{v} + \nabla p = \underset{\sim}{J}\times\underset{\sim}{B} \tag{23}$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla\times(\eta\underset{\sim}{J}-\underset{\sim}{v}\times\underset{\sim}{B}) = 0 \tag{24}$$

$$\nabla\cdot\underset{\sim}{v} = 0\ , \qquad \rho = \text{const.} \tag{25}$$

where η is a "small" constant scalar resistivity, and $\underset{\sim}{J} \equiv \nabla\times\underset{\sim}{B}$ is the current density. The plasma is assumed to be confined and isolated inside a toroidal solid conductor, on which the boundary conditions

$$\underset{\sim}{B}\cdot\hat{\underset{\sim}{n}} = 0\ , \qquad \underset{\sim}{v}\cdot\hat{\underset{\sim}{n}} = 0\ , \tag{26}$$

hold, where $\hat{\underset{\sim}{n}}$ is normal to the boundary. Similar to the previous section, we consider a slowly varying mean state, but this time containing only diffusion-induced flows, upon which is superimposed low-level turbulence. More precisely, and using previous notations, we use the following assumptions: (1) for the mean state, $a|\underset{\sim}{v}_0| = O(\eta)$, $a^2\partial/\partial t = O(\eta)$, $a|\nabla| = O(1)$; (2) for the fluctuations, $a^2\partial/\partial t >> O(\eta)$, $a|\nabla| >> O(1)$; and (3) the energy of the fluctuations, be it magnetic, kinetic or thermal, is smaller than the mean magnetic energy, i.e.,

$|\underset{\sim}{B}_1|^2$, $\rho|\underset{\sim}{v}_1|^2$, $p_1 << |\underset{\sim}{B}_o|^2$. Since $\underset{\sim}{J}_1 = \nabla\times\underset{\sim}{B}_1$, the fluctuating current may be large. We allow it to be of the same order as $\underset{\sim}{J}_o$. In the following, we will consider our variables to have been non-dimensionalized with respect to typical mean magnetic field parameters.

Averaging Eqs. (23) and (24), we get

$$\nabla p_o = \underset{\sim}{J}_o\times\underset{\sim}{B}_o \ , \tag{27}$$

$$\frac{\partial\underset{\sim}{B}_o}{\partial t} + \nabla\times(\eta\underset{\sim}{J}_o - \underset{\sim}{v}_o\times\underset{\sim}{B}_o - \langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle) = 0 \ , \tag{28}$$

where we note that Eq. (27) is only correct to 0(1), and implies that $\underset{\sim}{B}_o$ is in static equilibrium. Equation (28) then requires the evolution to take place through a sequence of equilibrium states. According to the Grad-Hogan theory,[20] the velocity $\underset{\sim}{v}_o$ is determined (actually, adjusts itself) so that this requirement is satisfied. It will be assumed in the following that $\underset{\sim}{B}_o$ forms a family of nested toroidal magnetic flux surfaces.

We now derive some properties of the turbulence-induced electric field, $\underset{\sim}{\varepsilon} \equiv \langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle$. The following relations must hold to order η,

$$\int \underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o d\tau = -\int \eta\langle\underset{\sim}{J}_1^2\rangle d\tau \ , \qquad \int \underset{\sim}{\varepsilon}\cdot\underset{\sim}{B}_o d\tau = 0 \ , \tag{29}$$

where the integrals are taken over the plasma volume. The first relation may be shown by averaging the energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\underset{\sim}{B}^2 + \frac{1}{2}\rho \underset{\sim}{v}^2\right) + \nabla\cdot\left[\frac{1}{2}\rho \underset{\sim}{v}^2\underset{\sim}{v} + p\underset{\sim}{v} + (\eta\underset{\sim}{J} - \underset{\sim}{v}\times\underset{\sim}{B})\times\underset{\sim}{B}\right] + \eta\underset{\sim}{J}^2 = 0\ , \qquad (30)$$

which may be obtained from Eqs. (23)-(25). The average of Eq. (30) yields, to O(η),

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\underset{\sim}{B}_o^2\right) + \eta\underset{\sim}{J}_o^2 + \eta\langle\underset{\sim}{J}_1^2\rangle + \nabla\cdot\underset{\sim}{c}_1 = 0\ , \qquad (31)$$

where $\underset{\sim}{c}_1$ represents energy flux. Another equation for the mean magnetic energy may be obtained from Eq. (28). The result is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\underset{\sim}{B}_o^2\right) + \eta\underset{\sim}{J}_o^2 - \underset{\sim}{\varepsilon}\cdot\underset{\sim}{J}_o + \nabla\cdot\underset{\sim}{c}_2 = 0 \qquad (32)$$

The surface integral $\int(\underset{\sim}{c}_1-\underset{\sim}{c}_2)\cdot\hat{\underset{\sim}{n}}dS$ represents the energy flux due to the fluctuations which, for an isolated plasma, vanishes. A comparison of Eqs. (31) and (32) yields the desired property of $\varepsilon\cdot\underset{\sim}{J}_o$.

The second property in Eq. (29) may be proved similarly by averaging the helicity equation (with $\underset{\sim}{B} = \nabla\times\underset{\sim}{A}$)

$$\frac{\partial}{\partial t}(\underset{\sim}{A}\cdot\underset{\sim}{B}) + 2\eta\underset{\sim}{J}\cdot\underset{\sim}{B} + \nabla\cdot\left[2(\eta\underset{\sim}{J}-\underset{\sim}{v}\times\underset{\sim}{B})\times\underset{\sim}{A} + \frac{\partial\underset{\sim}{A}}{\partial t}\times\underset{\sim}{A}\right] = 0\ . \qquad (33)$$

Averaging yields, to O(η),

$$\frac{\partial}{\partial t}(\underset{\sim}{A}_o\cdot\underset{\sim}{B}_o) + 2\eta\underset{\sim}{J}_o\cdot\underset{\sim}{B}_o + \nabla\cdot\underset{\sim}{c}_3 = 0\ , \qquad (34)$$

where $\underset{\sim}{c}_3$ represents helicity flux. However, from Eq. (28) we can write

$$\frac{\partial}{\partial t}(\underset{\sim}{A}_o\cdot\underset{\sim}{B}_o) + 2\eta\underset{\sim}{J}_o\cdot\underset{\sim}{B}_o - 2\underset{\sim}{\varepsilon}\cdot\underset{\sim}{B}_o + \nabla\cdot\underset{\sim}{c}_4 = 0 . \tag{35}$$

$\int(\underset{\sim}{c}_3-\underset{\sim}{c}_4)\cdot\hat{\underset{\sim}{n}}dS$ represents the helicity flux due to the fluctuations, which again vanishes. Comparing Eqs. (34) and (35) yields the form

$$\underset{\sim}{\varepsilon}\cdot\underset{\sim}{B}_o = \nabla\cdot\underset{\sim}{\xi} , \tag{36}$$

where $\underset{\sim}{\varepsilon}$ has zero net flux across the boundary, and the desired result follows.

From the two properties expressed in Eq. (29) we can guess the form of the component of $\underset{\sim}{\varepsilon}$ parallel to $\underset{\sim}{B}_o$, along the same lines used in the previous section. For a low-pressure plasma (which is the usual experimental situation), Eq. (27) implies that $\underset{\sim}{J}_o$ is nearly parallel to $\underset{\sim}{B}_o$. Writing $\underset{\sim}{J}_o = \lambda\underset{\sim}{B}_o$ and using Eq. (36) and the first relation of Eq. (29), we get after integration by parts

$$\int \underset{\sim}{\xi}\cdot\nabla\lambda d\tau \geq 0 . \tag{37}$$

This is analogous to Eq. (9) and similarly suggests, if the correlation of fluctuations across flux surfaces is negligible, that

$$\oint_V \xi_n dS = K^2\frac{d\lambda}{dV} , \tag{38}$$

where here the volume V labels flux surfaces and, again, $K^2 \geq 0$. Notice that $\nabla \cdot \underline{J}_o = 0$ implies that λ is nearly constant on flux surfaces.

Relation (38) was derived in Refs. 13 and 14 directly from the dynamical equations of resistive tearing modes.[4] It is important to notice that, when substituted in Eq. (28), it enters as a diffusive term of fourth order in spatial derivatives of $\underline{B}_o$ (since λ, the parallel current, is itself related to derivatives of $\underline{B}_o$). In this respect, it is significantly different from the eddy viscosity term derived before. Indeed, it can explain the phenomenon of plasma relaxation to a near Taylor state,[10,11] which cannot be explained by a simple enhancement of classical resistivity.[22]

As an illustration, we present here a numerical calculation of a steady state solution of a cylindrical plasma under the influence of low level turbulence and an externally applied electric field in the z direction (which is the plasma axis). We solve

$$\eta \underline{J} - \underline{\varepsilon} = E\hat{\underline{z}} \ , \qquad (39)$$

where the constant E is the applied axial voltage per unit length, and $\underline{\varepsilon}$ has the form

$$\underline{\varepsilon} = \alpha_o \frac{1}{rB^2} (\eta a^2 r B^2 \lambda')' \underline{B} + \beta_o \frac{\eta p'}{B^2} \hat{\underline{r}} \times \underline{B} \ . \qquad (40)$$

Here all quantities are mean field quantities, primes denote d/dr, a is the plasma radius, α_0 and β_0 are constants, and the pressure p is obtained from the equilibrium relation

$$\frac{d}{dr}\left(p + \frac{1}{2}\underset{\sim}{B}^2\right) + \frac{B_\theta^2}{r} = 0 \,, \qquad (41)$$

with p(a) = 0. The boundary condition $d\lambda/dr = 0$ at r = a is used to conform with properties (29).

Expression (40) was derived in Ref. 14 under the assumption that tearing modes[4] are mainly responsible for the component of $\underset{\sim}{\varepsilon}$ parallel to $\underset{\sim}{B}$, and interchange modes[4] are mainly responsible for the perpendicular component. The expression given here for the latter is, in fact, new and corresponds to a version of the mode dynamics appropriate for low-pressure plasmas.[23] Interestingly, this component of $\underset{\sim}{\varepsilon}$ equals $-\beta_0\eta$ times the perpendicular component of $\underset{\sim}{J}$. Thus, it corresponds to a simple enhancement of the perpendicular resistivity.

Equations (39)-(41) are solved after non-dimensionalization. We define the volume-averaged B_z to be

$$\bar{B}_z = \frac{2}{a^2}\int_0^a rB_z dr \,, \qquad (42)$$

and scale $\underset{\sim}{B}$ by $\bar{B}_z$, p by $\bar{B}_z^2$, and E by $\bar{B}_z\eta/a$. In addition, r is normalized to a. Effectively, we now have a = 1, η = 1, $0 \le r \le 1$, with boundary conditions $\lambda'(1) = 0$ and $\bar{B}_z = 1$. In the calculation presented in Fig. 2, we take $\alpha_0 = 1$, as has been estimated for RFP plasmas,[14] and E = 12, which also corresponds to typical machine parameters.[14] β_o, here taken to equal 100, was chosen large enough to yield a sufficiently low plasma pressure. As to the results, Fig. 2a shows the celebrated field reversal, which is not possible if $\alpha_o = 0$, that is, without effect of the parallel eddy resistivity. Figure 2b demonstrates that the equilibrium is indeed approximately a Taylor state, with a very flat $\lambda(r)$ profile.

IV. TURBULENT DIFFUSION IN COMPRESSIBLE PLASMAS

Extension of the previous analysis to the case of a compressible plasma is not difficult, but it requires a redefinition of $\boldsymbol{\epsilon}$. Such a device has already been used in Ref. 14. However, the guiding consideration given there is inappropriate if the plasma has non-vanishing diffusion coefficients other than the resistivity, a case not treated in Ref. 14. This situation is corrected in the present section, and we are able to deal with a compressible plasma subject to the effects of heat conduction and viscosity, as well as resistivity. The plasma evolves according to the following equations:

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} + \nabla p = \mathbf{J}\times\mathbf{B} + \nabla\cdot(\mu\boldsymbol{\sigma}) \tag{43}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\times(\eta\mathbf{J} - \mathbf{v}\times\mathbf{B}) = 0 \tag{44}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{45}$$

$$\frac{\partial}{\partial t}(\rho s) + \nabla\cdot(\rho s\mathbf{v}) = \frac{\gamma-1}{2T}\,\mu\,\mathrm{tr}\,(\boldsymbol{\sigma}^2) + \frac{\gamma-1}{T}\,\eta\mathbf{J}\cdot\mathbf{J} + (\gamma-1)\nabla\cdot\left(\frac{1}{T}\,\kappa\nabla T\right) + \frac{\gamma-1}{T^2}\,\kappa\nabla T\cdot\nabla T\ , \tag{46}$$

where the specific entropy s satisfies the equation of state $s = \log\,(p/\rho^{\gamma})$. Here $T = p/\rho$ is the temperature, γ is a constant, $\boldsymbol{\sigma}$ is the rate-of-strain tensor with components

$$\sigma_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\nabla\cdot\mathbf{v}\ ,$$

and tr indicates the trace of the tensor. Note that the last two terms of Eq. (46) are written in a way which enables easy averaging.

We consider here the case of a tensor resistivity η (assumed to be small), and a tensor heat conductivity κ, while, for simplicity, the viscosity μ is taken to be a scalar function. All the diffusion coefficients are allowed to depend on $\underline{B}$, ρ, and T, but not on their derivatives (which is the physical situation). We assume now that $\underline{J}_1$ and ∇T_1 are at most O(1), that κ is of the same order as η, and that μ is so small that $\mu|\nabla v_{1i}|^2$ is at most $O(\eta)$, for any component i of the fluctuating velocity. These assumptions imply that the fluctuating parts of η, κ and μ are too small to enter the averaged equation. Thus, whenever the diffusion coefficients are used in the following, they should be taken as evaluated for the mean fields. We follow again the general ordering scheme spelled out in the previous section, with the added requirement that $|\rho_1| << \rho_o$, except that $\underline{v}_o$ is allowed to be as large as $\underline{v}_1$.

While averaging equations, it is desirable to average them in their conservation form for then the averaging operation is carried out under the derivatives and the order of the various terms is transparent. The average of the momentum conservation equation, Eq. (43) combined with (45), yields again to O(1)

$$\nabla p_o = \underline{J}_o \times \underline{B}_o \ , \qquad (47)$$

while the averages of Eqs. (44)-(46) yield to $O(\eta)$,

$$\frac{\partial \underset{\sim}{B}_o}{\partial t} + \nabla\times(\eta\underset{\sim}{J}_o - \underset{\sim}{v}_o\times\underset{\sim}{B}_o - \langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle) = 0\ , \tag{48}$$

$$\frac{\partial \rho_o}{\partial t} + \nabla\cdot(\rho_o\underset{\sim}{v}_o + \langle\rho_1\underset{\sim}{v}_1\rangle) = 0\ , \tag{49}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\rho_o s_o) &+ \nabla\cdot(\rho_o s_o\underset{\sim}{v}_o + \rho_o\langle s_1\underset{\sim}{v}_1\rangle + s_o\langle\rho_1\underset{\sim}{v}_1\rangle)\\ &= \frac{\gamma-1}{2T_o}\,\mu\ \mathrm{tr}\ \langle\underset{=}{\sigma}^2\rangle + \frac{\gamma-1}{T_o}(\eta\underset{\sim}{J}_o\cdot\underset{\sim}{J}_o + \langle\eta\underset{\sim}{J}_1\cdot\underset{\sim}{J}_1\rangle)\\ &\quad + (\gamma-1)\nabla\cdot\left(\frac{1}{T_o}\,\kappa\nabla T_o\right) + \frac{\gamma-1}{T_o^2}(\kappa\nabla T_o\cdot\nabla T_o + \langle\kappa\nabla T_1\cdot\nabla T_1\rangle)\ ,\end{aligned} \tag{50}$$

where $\langle\rho_1 s_1\underset{\sim}{v}\rangle$ was dropped from Eq. (50) as too small. Using Eqs. (43)-(46) we form the energy equation

$$\begin{aligned}\frac{\partial}{\partial t}&\left(\frac{1}{2}\underset{\sim}{B}^2 + \frac{1}{2}\rho\underset{\sim}{v}^2 + \frac{p}{\gamma-1}\right)\\ &+ \nabla\cdot\left[(\eta\underset{\sim}{J}-\underset{\sim}{v}\times\underset{\sim}{B})\times\underset{\sim}{B} + \frac{1}{2}\rho\underset{\sim}{v}^2\underset{\sim}{v} + \frac{\gamma}{\gamma-1}\,p\underset{\sim}{v} - \kappa\nabla T - \mu\underset{=}{\sigma}\underset{\sim}{v}\right] = 0\ .\end{aligned} \tag{51}$$

Averaging it yields to $O(\eta)$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\underset{\sim}{B}_o^2 + \frac{p_o}{\gamma-1}\right) &+ \nabla\cdot\left[(\eta\underset{\sim}{J}_o - \underset{\sim}{v}_o\times\underset{\sim}{B}_o - \langle\underset{\sim}{v}_1\times\underset{\sim}{B}_1\rangle)\times\underset{\sim}{B}_o + \frac{1}{2}\rho_o\langle\underset{\sim}{v}^2\underset{\sim}{v}\rangle\right.\\ &\left.+ \frac{\gamma}{\gamma-1}(p_o\underset{\sim}{v}_o + \langle p_1\underset{\sim}{v}_1\rangle) - \langle(\underset{\sim}{v}_1\times\underset{\sim}{B}_o)\times\underset{\sim}{B}_1\rangle - \kappa\nabla T_o\right] = 0\ ,\end{aligned} \tag{52}$$

where we still use $\underset{\sim}{v} = \underset{\sim}{v}_o + \underset{\sim}{v}_1$, and a term of order $\rho_1|\underset{\sim}{v}|^3$ is dropped as too small. The term $(\underset{\sim}{v}\times\underset{\sim}{B}_1)\times\underset{\sim}{B}_1$ is also dropped for the same reason.

Comparing Eq. (52) with an expression for $\partial\left(\underline{B}_0^2/2 + p_0/(\gamma-1)\right)/\partial t$ based on (47)-(50), we find

$$\langle\underline{v}_1\times\underline{B}_1\rangle\cdot\underline{J}_0 + \frac{1}{(\gamma-1)T_0}\langle p_1\underline{v}_1\rangle\cdot\nabla T_0 - \frac{T_0}{\gamma-1}\langle\rho_1\underline{v}_1\rangle\cdot\nabla s_0$$

$$= -\frac{\mu}{2}\,\mathrm{tr}\,\langle\underline{\underline{\sigma}}_1^2\rangle - \langle\eta\underline{J}_1\cdot\underline{J}_1\rangle - \frac{1}{T_0}\langle\kappa\nabla T_1\cdot\nabla T_1\rangle + f\ , \qquad (53)$$

where we have used the relation (to the relevant order) $s_1 = p_1/p_0 - \gamma\rho_1/\rho_0$, and where

$$f = -\nabla\cdot\langle(p_1 + \underline{B}_0\cdot\underline{B}_1)\underline{v}_1 - (\underline{v}_1\cdot\underline{B}_1)\underline{B}_0 + \frac{1}{2}\rho_0\underline{v}^2\underline{v}\rangle\ . \qquad (54)$$

The function f appeared in Ref. 14, where it was argued that, if any, only the last term of (54) which expresses kinetic energy flux, may be large enough to enter the mean equations. The quantity $p_1 + \underline{B}_0\cdot\underline{B}_1$, the fluctuating total pressure, is smaller than $\underline{B}_1$ for small-scale motins, while the second term does not involve flux across the mean magnetic flux surfaces.

We can simplify the mean equations considerably by using the following quantities (which will be discussed at the end of this section),

$$\underline{\varepsilon} = \langle\underline{v}_1\times\underline{B}_1\rangle - \frac{1}{\rho_0}\langle\rho_1\underline{v}_1\rangle\times\underline{B}_0 \qquad (55)$$

$$\underset{\sim}{u}_0 = \underset{\sim}{v}_0 + \frac{1}{\rho_0} \langle \rho_1 \underset{\sim}{v}_1 \rangle \ . \qquad (56)$$

The evolution of mean quantities is now determined by Eq. (47), as well as by

$$\frac{\partial \underset{\sim}{B}_0}{\partial t} + \nabla \times (\eta \underset{\sim}{J}_0 - \underset{\sim}{\varepsilon} - \underset{\sim}{u}_0 \times \underset{\sim}{B}_0) = 0 \qquad (57)$$

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \underset{\sim}{u}_0) = 0 \qquad (58)$$

$$\frac{\partial p_0}{\partial t} + \underset{\sim}{u}_0 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underset{\sim}{u}_0 = (\gamma - 1)\left[\eta \underset{\sim}{J}_0 \cdot \underset{\sim}{J}_0 - \underset{\sim}{\varepsilon} \cdot \underset{\sim}{J}_0 + \nabla \cdot (\kappa \nabla T_0) - \nabla \cdot \left(\frac{p_0}{\gamma - 1} \langle s_1 \underset{\sim}{v}_1 \rangle\right) + f\right] \qquad (59)$$

The relation $p_0 = \rho_0 T_0$ also holds to $O(1)$, which is the required order for Eqs. (57)-(59). Equation (53) now takes the form

$$\underset{\sim}{\varepsilon} \cdot \underset{\sim}{J}_0 + \frac{1}{\gamma - 1} \rho_0 \langle s_1 \underset{\sim}{v}_1 \rangle \cdot \nabla T_0 = - \left\{\frac{\mu}{2} \, \mathrm{tr} \, \langle \underset{\approx}{\sigma}_1^2 \rangle + \langle \eta \underset{\sim}{J}_1 \cdot \underset{\sim}{J}_1 \rangle + \frac{1}{T_0} \langle \kappa \nabla T_1 \cdot \nabla T_1 \rangle \right\} + f \ , \qquad (60)$$

and expresses heat production by the fluctuations. In fact, this form[24] suggests that $\underset{\sim}{\varepsilon}$ is the turbulent diffusion flux associated with magnetic field gradients (the current), and $\langle s_1 \underset{\sim}{v}_1 \rangle$ is the diffusion flux associated with ∇T_0, namely, the turbulent heat conduction term. This interpretation is supported by the result, proven in Ref. 14 [along the same lines which lead from Eq. (62) to Eq. (65) in this paper], that $\langle s_1 \underset{\sim}{v}_1 \rangle$ vanishes when $\kappa \to 0$ so that $\underset{\sim}{\varepsilon}$ is generated by motion due to resistive instabilities, and by the result, to be proven in the next section, that $\underline{\varepsilon}$ vanishes as $\eta \to 0$.

Relation (60) can be used to guess the form of the parallel component of $\underline{\varepsilon}$, as was done in the previous section. For a low pressure plasma $\underline{J}_0 \approx \lambda \underline{B}_0$ and $\rho_0 \nabla T_0$ is small. Thus, Eq. (60) implies $\int \lambda \underline{\varepsilon} \cdot \underline{B}_0 d^3x \leq 0$. Equation (36) remains valid here as well, because redefinition (55) does not affect the parallel component of $\underline{\varepsilon}$. Using the argument given in Section III we end up with exactly the same form as in the incompressible case, namely, Eq. (38).

Finally, we discuss redefinitions (55) and (56), which brought about a significant simplification of the evolution equations (57)-(59). The effect of the small scales enters (in addition to f, perhaps) in the form of only three magnetic surface integrals on each surface; two associated with surface components of $\underline{\varepsilon}$, and the integral of $\oint_V \langle s_1 v_{1n} \rangle dS$. This is a consequence of the Grad-Hogan theory[20,21] when applied to our equations. The small diffusion velocity of order η is now $\underline{u}_0$, and not $\underline{v}_0$ which may be cancelled by the $\langle \rho_1 \underline{v}_1 \rangle$ term in Eq. (56). This term acts as a Gallilean shift in the frame of reference, and the redefined $\underline{\varepsilon}$ is related to $\langle \underline{v}_1 \times \underline{B}_1 \rangle$ through the transformation governing the electric field when observed from a moving frame.

The physical meaning of $\underline{u}_0$ is best seen within the context of kinetic theory. Let $f(\underline{x},\underline{w},t)$ be the distribution function of the plasma, where $\underline{w}$ is the microscopic velocity variable, and $\int f d^3\underline{w} = n$, the number density. The macroscopic velocity $\underline{v}$ is given by $n\underline{v} = \int \underline{w} f d^3\underline{w}$. Let us write f as the sum of mean and fluctuating parts,

$f = f_o + f_1$. Integrating over velocity space we get $n = n_o + n_1$ where, clearly, $n_o = \int f_o d^3\underset{\sim}{w}$. The next moment yields

$$n\underset{\sim}{v} = n_o\underset{\sim}{u}_o + \int \underset{\sim}{w} f_1 d^3\underset{\sim}{w} \ , \qquad (61)$$

where, by definition, $\underset{\sim}{u}_o = n_o^{-1}\int \underset{\sim}{w} f_o d^3\underset{\sim}{w}$. The last term in Eq. (61) has zero mean, and we have $n_o\underset{\sim}{u}_o = \langle n\underset{\sim}{v}\rangle$. This coincides with the definition of $\underset{\sim}{u}_o$ in Eq. (56), and establishes $\underset{\sim}{u}_o$ as the natural macroscopic velocity associated with the mean distribution function f_o.

V. TURBULENCE WITH ZERO RESISTIVITY

The effect of the turbulent heat conductivity is seen most clearly when the plasma resistivity vanishes. This is the case we now investigate. The same scaling as in the previous section is assumed, where now the small parameter is attributable either to κ (which does not appear explicitly in the following calculations) or to any other cause of the slow diffusion of mean fields. We will show, in this case, that $\underset{\sim}{\varepsilon}$ defined by Eq. (55) vanishes, at least to the order of $|\underset{\sim}{v}_1| \times |\underset{\sim}{B}_1|$.

First, we consider the diffusion of a scalar function ψ which is advected by the fluid,

$$\frac{\partial \psi}{\partial t} + \underset{\sim}{v} \cdot \nabla \psi = 0 \ . \tag{62}$$

Using mass conservation, we form the conservation equation $\partial(\rho\psi)/\partial t + \nabla \cdot (\rho\psi\underset{\sim}{v}) = 0$. Averaging yields, as in previous such exercises,

$$\frac{\partial}{\partial t} (\rho_o \psi_o) + \nabla \cdot (\rho_o \psi_o \underset{\sim}{u}_o + \rho_o \langle \psi_1 \underset{\sim}{v}_1 \rangle) = 0 \ , \tag{63}$$

where $\langle \rho_1 \psi_1 \underset{\sim}{v} \rangle$ is dropped as too small. In Eq. (63), $\underset{\sim}{u}_o$ is defined exactly as in Eq. (56). We now notice that any function $F(\psi)$ must also satisfy Eq. (62), as well as an equation corresponding to (63). Writing $F(\psi_o + \psi_1) = F(\psi_o) + \dot{F}(\psi_o)\psi_1 + O(\psi_1^2)$, where the dot indicates a derivative, we find that, up to an error of order ψ_1^2, $F_o = F(\psi_o)$,

$F_1 = \dot{F}(\psi_o)\psi_1$. Thus, Eq. (63) written for F takes the form (correct to the order of $|\psi_1 \underset{\sim}{v}_1|$),

$$\frac{\partial}{\partial t}(\rho_o F_o) + \nabla\cdot[\rho_o F_o \underset{\sim}{u}_o + \rho_o \dot{F}(\psi_o)\langle\psi_1 \underset{\sim}{v}_1\rangle] = 0 \ . \qquad (64)$$

Multiplying Eq. (63) by $\dot{F}(\psi_o)$ and comparing it with Eq. (64), we get, after using Eq. (58),

$$\langle\psi_1 \underset{\sim}{v}_1\rangle\cdot\nabla\dot{F}(\psi_o) = 0 \ . \qquad (65)$$

Thus, $\langle\psi_1 \underset{\sim}{v}_1\rangle$ is perpendicular to constant-ψ_o surfaces. We now identify ψ_o with the flux function of $\underset{\sim}{B}_o$ that labels its toroidal flux surfaces. Indeed, the average of $\nabla\cdot(\psi \underset{\sim}{B}) = 0$ yields, to order 1, $\nabla\cdot(\psi_o \underset{\sim}{B}_o) = 0$ which implies $\underset{\sim}{B}_o\cdot\nabla\psi_o = 0$. In all the following discussion, only the component of $\langle\psi_1 \underset{\sim}{v}_1\rangle$ normal to ψ_o-surfaces appears (which is a consequence of the Grad-Hogan theory[20,21]). We may thus view Eq. (65) as implying $\langle\psi_1 \underset{\sim}{v}_1\rangle = 0$, and Eq. (63) reads

$$\frac{\partial\psi_o}{\partial t} + \underset{\sim}{u}_o\cdot\nabla\psi_o = 0 \ . \qquad (66)$$

Consider now the turbulent diffusion of $\underset{\sim}{B}$, which obeys the "frozen-in" equation

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla\times(\underset{\sim}{B}\times\underset{\sim}{v}) = 0 \ . \qquad (67)$$

The field $\underset{\sim}{b} \equiv \psi \underset{\sim}{B}$ satisfies the same equation, and its average (correct to the order of $|\underset{\sim}{v}_1 \underset{\sim}{B}_1|$) yields

$$\frac{\partial}{\partial t} (\psi_o \underset{\sim}{B}_o) + \nabla \times [\psi_o (\underset{\sim}{B}_o \times \underset{\sim}{u}_o - \underset{\sim}{\varepsilon})] = 0 \;, \qquad (68)$$

Using Eqs. (57) and (66), we find $\underset{\sim}{\varepsilon} \times \nabla \psi_o = 0$. Since only the components of $\underset{\sim}{\varepsilon}$ tangential to ψ_o-surfaces enter the equations, we may conclude that $\underset{\sim}{\varepsilon} = 0$ as claimed.

Returning to Eq. (60), we find

$$\int \rho_o \langle s_1 \underset{\sim}{v}_1 \rangle \cdot \nabla T_o d^3x \leq 0 \;, \qquad (69)$$

where the integral is taken over the volume of the plasma. As in the previous sections, low-level turbulence with negligible correlation accross flux surfaces implies

$$\oint_V \langle s_1 v_{1n} \rangle dS = -K^2 \frac{dT_o}{dV} \qquad (70)$$

on every mean magnetic surface, where K^2 is a non-negative function. This relation suggests that, in the case of zero resistivity, turbulent heat conductivity acts in a similar way to classical molecular heat conductivity. This conclusion may have to be modified if the resistivity is finite, but this question will not be pursued here.

VI. CONCLUSION

In this article we have presented a dynamical description of MHD plasmas affected by low-level turbulence. Two-dimensional fluid flow was also discussed as a simple example. While, except for the fluid case, we never write down the dynamical equations of the fluctuations, a comparison of conservation laws for the full variables and for the mean variables makes it possible to derive important properties of the turbulent diffusion terms. We thus find that turbulent resistivity is quite different from its molecular counterpart, and can explain spontaneous toroidal field reversal in RFP experiments.[9,10] Turbulent heat conductivity, on the other hand, is more similar to the molecular effect and transports heat in the direction opposite to the mean temperature gradient.

An issue which is left open here is the question of "cross effects"; the effect of temperature gradient and finite heat conductivity (which unleashes the corresponding instabilities) on the turbulent resistivity, and the effect of resistivity on the turbulent heat conductivity. The question that presents itself is whether there is some symmetry relation between these two effects, similar to the relation which exists between different classical diffusion effects according to Onsager's theorem.[24] Our expectation is that the answer to this question is in the affirmative, and base our opinion on the fact the Onsager relations apply to phenomenological diffusion laws, regardless of whether their origin is collisional or turbulent. A credible answer to this question, however, has to be based on the macroscopic (fluid) equations of the turbulent fluctuations.

ACKNOWLEDGMENTS

The author is indebted to Dr. M. Hossain for a number of helpful discussions on fluid turbulence, and to Professors A. Bhattacharjee and K. Imre for contributing the figures. To Professor H.R. Strauss is due the observation that eddy viscosity must have a functional form similar to that of the molecular viscosity.

This research was supported by U.S. Department of Energy Grant No. DE-FG02-86ER53223 and by the Solar Terrestrial Theory Program of the National Aeronautics and Space Administration under Grant No. NAGW-781.

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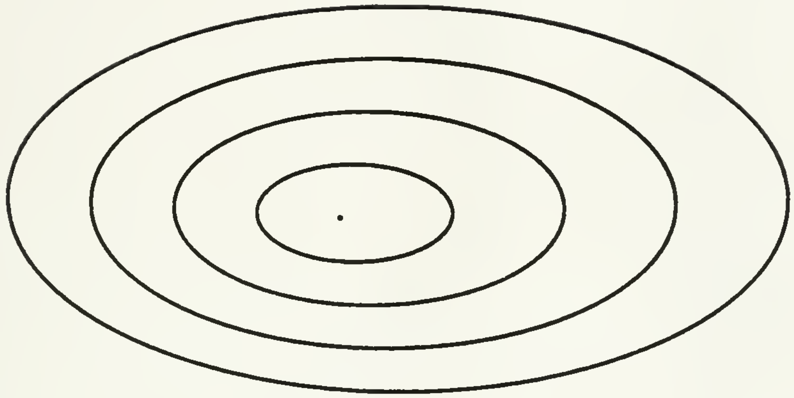

FIGURE 1

The geometry of nested streamlines

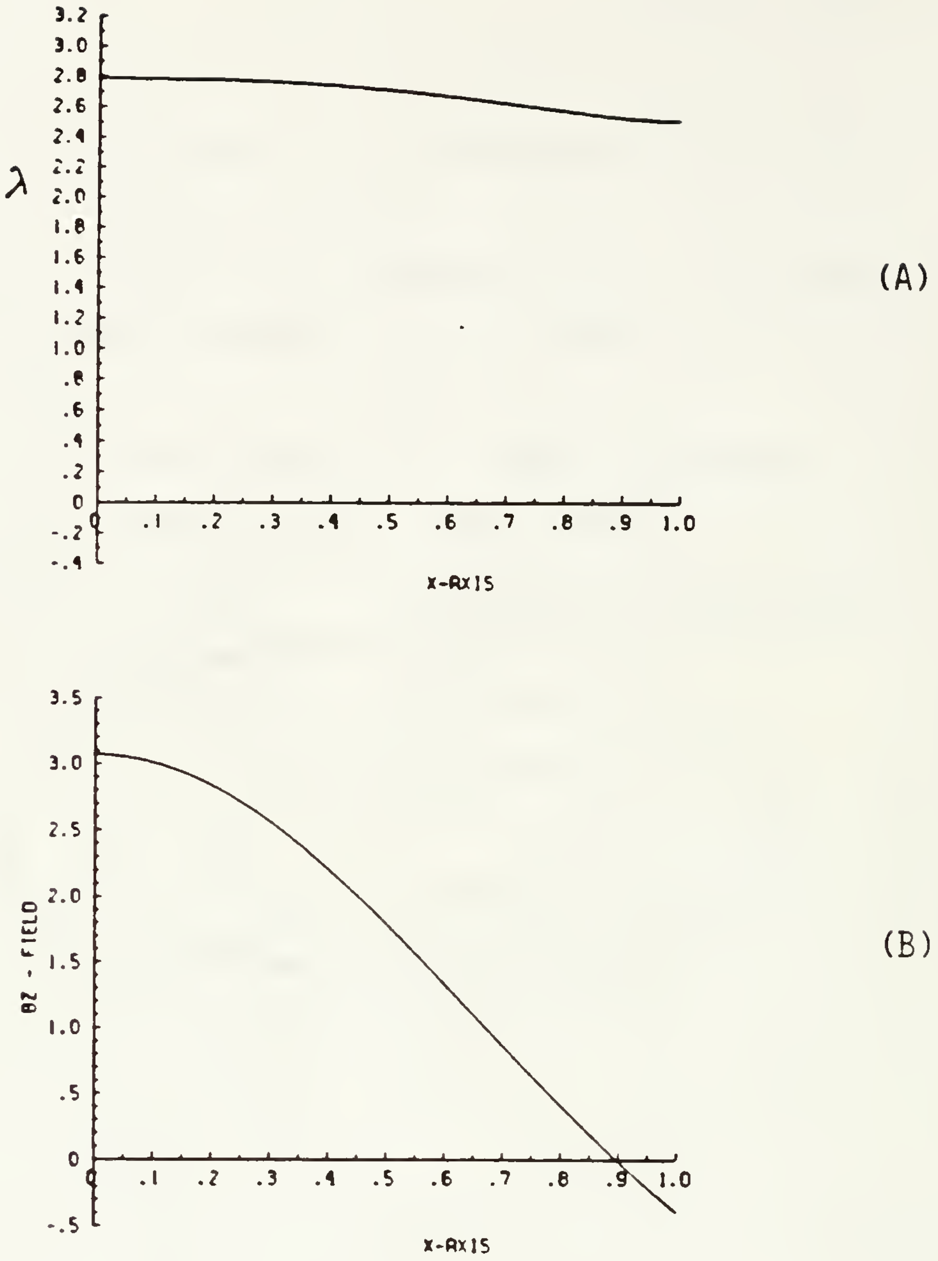

FIGURE 2

Radial profiles of (a), the toroidal field B_z and (b), the parallel current $\lambda = \underline{J}_0 \cdot \underline{B}_0 / |\underline{B}_0|^2$, are shown for $\alpha_0 = 1$, $\beta_0 = 100$, $E = 12$. Here $x = r/a$ is the normalized radial variable.

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